2016 여름학기 수학 및 연습 2

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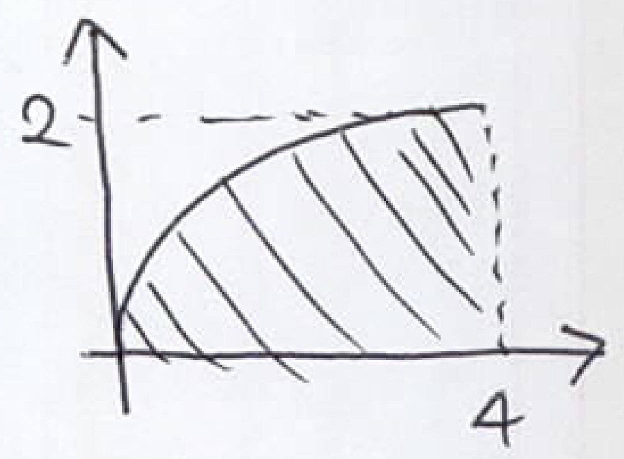
#1

(a)



$$\int_0^2\int_{y^2}^4 ye^{-x^2}\,dx\,dy = \int_0^4\int_0^{\sqrt{x}} ye^{-x^2}\,dy\,dx \quad \text{(by 푸비니 정리)}$$

10점

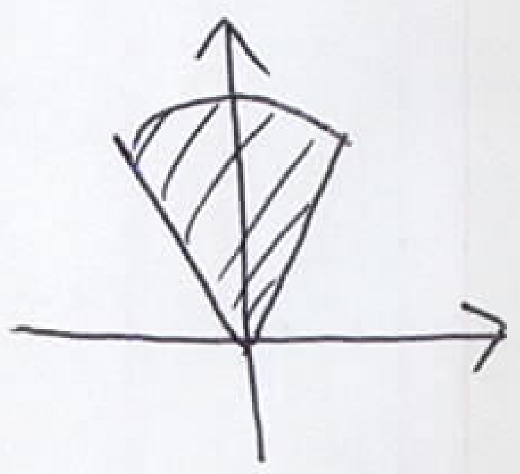
$$= \int_0^4 \frac{1}{2}xe^{-x^2}\,dx$$

$$= \frac{1}{4}\int_0^{16} e^{-u}\,du \quad \begin{pmatrix} u = x^2 \\ du = 2x\,dx \end{pmatrix}$$

$$= \frac{1}{4}(1-e^{-16})$$

15점.

(b)



$$\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{\sqrt{3}|x|}^{\sqrt{1-x^2}} (x^2+y^2+1)^{-\frac{5}{2}}\,dy\,dx$$

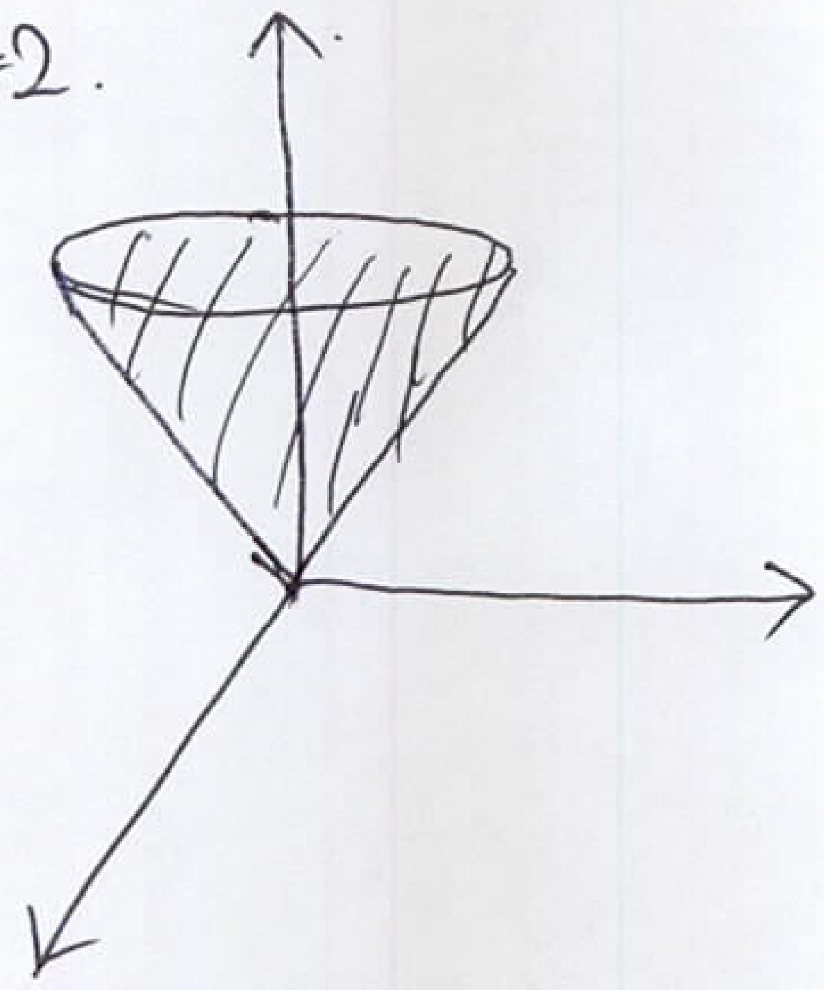
$$= \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}}\int_0^1 (r^2+1)^{-\frac{5}{2}}\cdot r\,dr\,d\theta$$

10점

$$= \frac{\pi}{3}\cdot\frac{1}{2}\left[-\frac{2}{3}(r^2+1)^{-\frac{3}{2}}\right]_0^1 = \frac{\pi}{9}\left(1-\frac{1}{2\sqrt{2}}\right)$$

15점

#2.



$(x,y,z) \longrightarrow (r,\theta,z)$: 원기둥 좌표계로 치환하자.

$\Rightarrow \; 0 \le \theta \le 2\pi \; , \; 0 \le z \le 2 \; , \; 0 \le r \le z.$ ┘5점

$$\iiint_D \sqrt{x^2+y^2} \, dx\,dy\,dz$$

$$= \int_0^{2\pi}\int_0^{2}\int_0^{z} r \cdot r\,dr\,dz\,d\theta$$ ┘10점

$$= 2\pi \int_0^2 \frac{1}{3} z^3 dz = 2\pi \cdot \frac{1}{3} \cdot \frac{16}{4} = \frac{8}{3}\pi.$$ ┘20점.

* 원기둥 좌표계 이외의 좌표계를 사용할 시, 위 기준에 따라 채점.

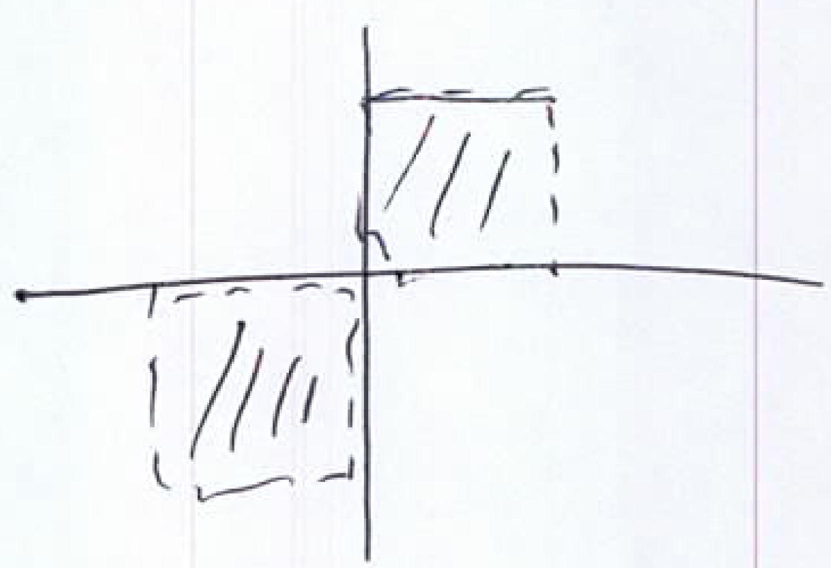
3.

(a) $\begin{aligned} x &= u^2 - v^2 \\ y &= 2uv \end{aligned}$ 이므로 $-1 + \frac{y^2}{4} < x < 1 - \frac{y^2}{4}$, $0 < y < 2$

$$\Downarrow$$

$$-1 + u^2v^2 < u^2 - v^2 < 1 - u^2v^2 , \quad 0 < uv < 1$$

$$\Longrightarrow \begin{cases} (u^2-1)(v^2+1) < 0 \\ (u^2+1)(v^2-1) < 0 \\ 0 < uv < 1 \end{cases} \Longrightarrow \quad |u| < 1 , \ |v| < 1 , \ 0 < uv < 1$$



가역함수가 되기 위해 $U = (0,1) \times (0,1)$ 혹은 $(-1,0) \times (-1,0)$ 으로 택한다.

채점기준 : U를 2개를 모두 택한 경우 5점. 하나 정확히 택하면 10점.

(b)

$$G'(u,v) = \begin{pmatrix} 2u & -2v \\ 2v & 2u \end{pmatrix} , \quad |\det G'(u,v)| = 4(u^2+v^2)$$

By 치환 적분법,

$$\iint_{W=G(U)} \sqrt{x^2+y^2}\, dx\,dy = \iint_U 4(u^2+v^2)^2\, du\,dv$$ 5점

$$= 4\int_0^1\int_0^1 (u^2+v^2)^2\, du\,dv$$

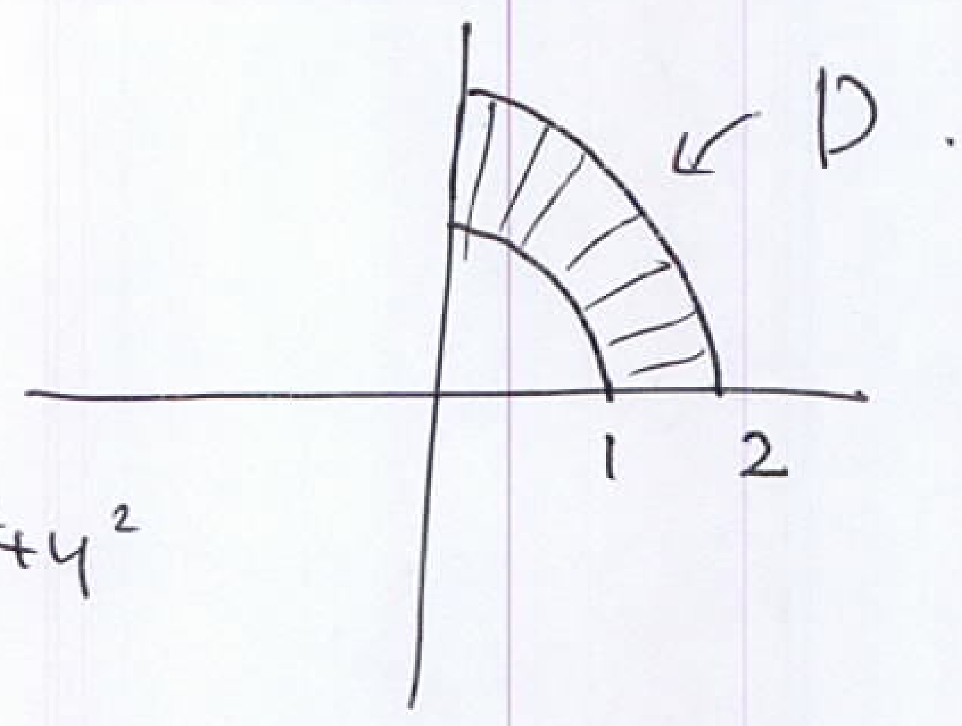
$$= \frac{112}{45}.$$

5점.

채점기준 : · U가 안맞아도 치환적분 정확히하면 5점.

· 영역을 2개로 적분하여 값이 2배면 5점.

4.



$$(\text{flux}) = \int_{\partial D} \mathbb{F} \cdot n \, ds \underset{\text{발산정리}}{=} \iint_D \operatorname{div} \mathbb{F} \, dV_2$$

$\operatorname{div} \mathbb{F} = x^2 + y^2$

10점.

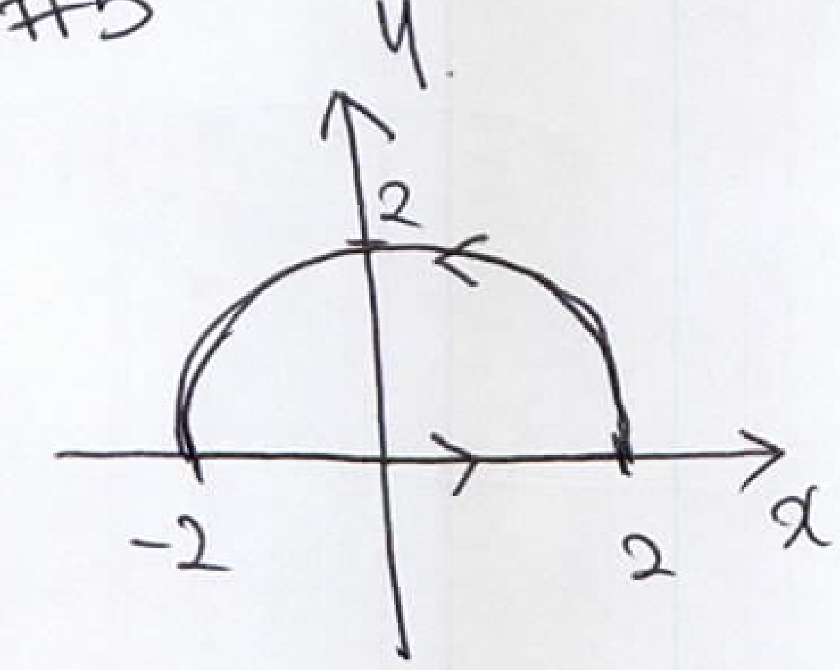
$$= \int_0^{\frac{\pi}{2}} \int_1^2 r^2 \cdot r \, dr \, d\theta$$

5점.

$$= \frac{15}{8} \pi$$

.5점.

#5



주어진 곡선을 C라고 하자.

곡선 C를 따라 $\mathbb{F}$의 일

$$= \int_C \mathbb{F} \cdot ds$$

$$= \iint_{\text{int } C} \text{rot } \mathbb{F} \, dxdy \quad (\text{by 그린정리})$$

┘5점

$$= \iint_{\text{int } C} (3x^2+3y^2) \, dxdy \quad (\text{rot } \mathbb{F} = 3x^2+3y^2)$$

┘10점

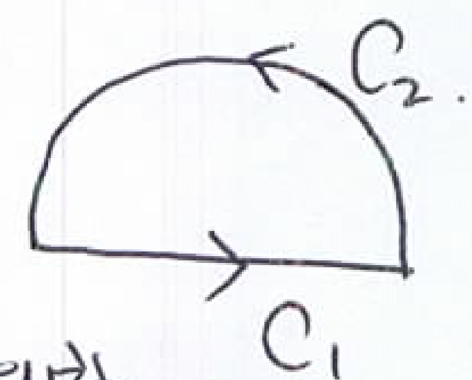
$$= 3\int_0^{\pi}\int_0^2 r^2 \, r dr d\theta$$

┘15점

$$= 12\pi$$

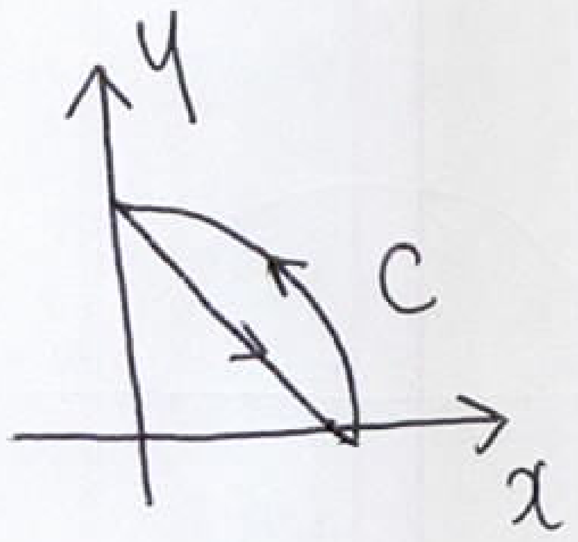
┘20점.

* 그린 정리를 사용하지 않고 직접구할 경우



C_1에 대한 계산 10점, C_2에 대한 계산 10점.

#6



$$\int_C (\arctan x - y^2)\,dx + (x^2+\sin y)\,dy$$

$$= \iint_{\text{int } C} \text{rot } \mathbb{F} \; dx\,dy \quad \text{(by 그린정리).} \quad \left(\mathbb{F}(x,y) = (\arctan x - y^2,\ x^2+\sin y)\right)$$

5점

$$= \iint_{\text{int } C} (2x+2y)\,dx\,dy$$

10점

$$= \int_0^1 \int_{1-x}^{\sqrt{1-x^2}} (2x+2y)\,dy\,dx$$

15점

$$= \int_0^1 \left[2xy+y^2\right]_{1-x}^{\sqrt{1-x^2}} dx$$

$$= \int_0^1 2x\sqrt{1-x^2}\,dx$$

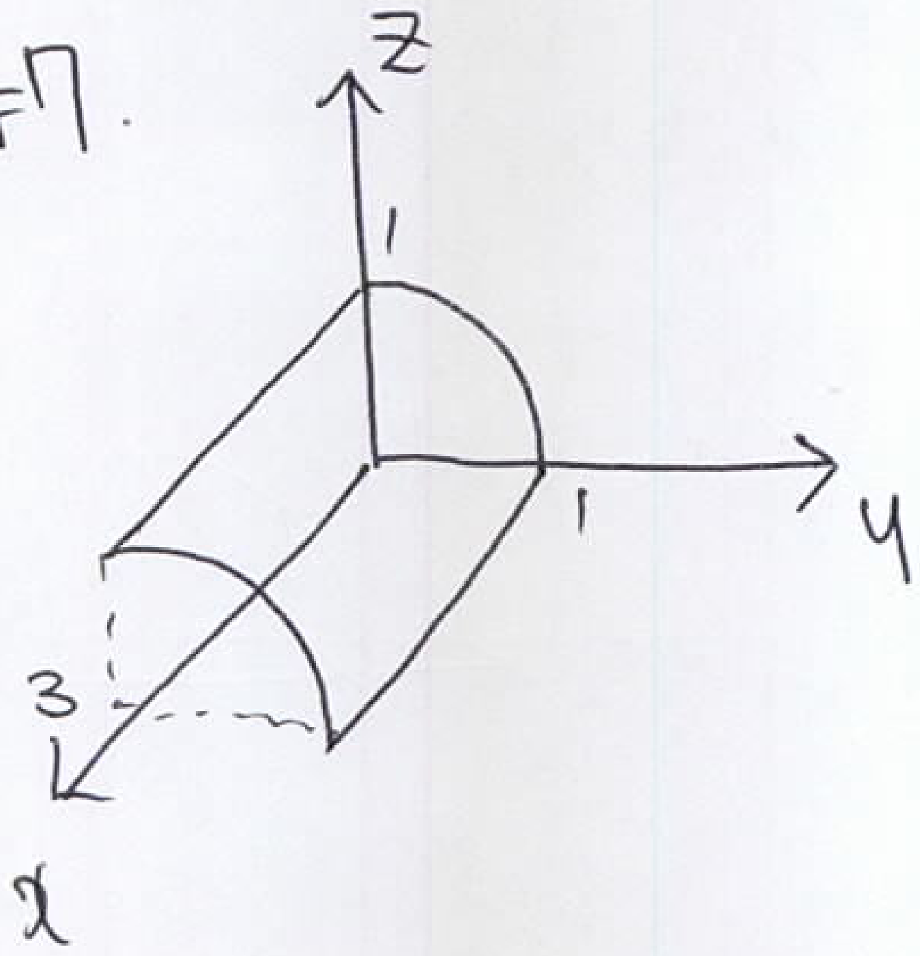
$$= -\frac{2}{3}\left[(1-x^2)^{\frac{3}{2}}\right]_0^1 = \frac{2}{3}$$

20점

* 위와 다른 방법으로 $\iint_{\text{int } C}(2x+2y)\,dx\,dy$ 를 계산할 경우,

위의 기준에 따라 채점.

#7.



곡면 S를 다음과 같이 매개화하자.

$$X(x,\theta) = (x, \cos\theta, \sin\theta), \quad 0 \le x \le 3, \quad 0 \le \theta \le \frac{\pi}{2}$$

$$X_x = (1, 0, 0)$$

$$X_\theta = (0, -\sin\theta, \cos\theta).$$

$$\Rightarrow |N| = |X_x \times X_\theta| = 1.$$

$$\text{Area}(S) = \iint_S dS = \int_0^{\frac{\pi}{2}} \int_0^3 1 \cdot dx\,d\theta = \frac{3}{2}\pi.$$

$$M = \iint_S f\,dS = \int_0^{\frac{\pi}{2}} \int_0^3 (x^2\cos\theta + \sin\theta)\,dx\,d\theta = 12.$$

$$\text{평균 밀도} = \frac{M}{\text{Area}(S)} = \frac{8}{\pi}$$

$$\bar{x} = \frac{1}{M}\iint_S xf\,dS = \frac{1}{12}\int_0^{\frac{\pi}{2}}\int_0^3 (x^3\cos\theta + x\sin\theta)\,dx\,d\theta = \frac{33}{16}$$

$$\bar{y} = \frac{1}{M}\iint_S yf\,dS = \frac{1}{12}\int_0^{\frac{\pi}{2}}\int_0^3 (x^2\cos^2\theta + \cos\theta\sin\theta)\,dx\,d\theta = \frac{3}{16}\pi + \frac{1}{8}$$

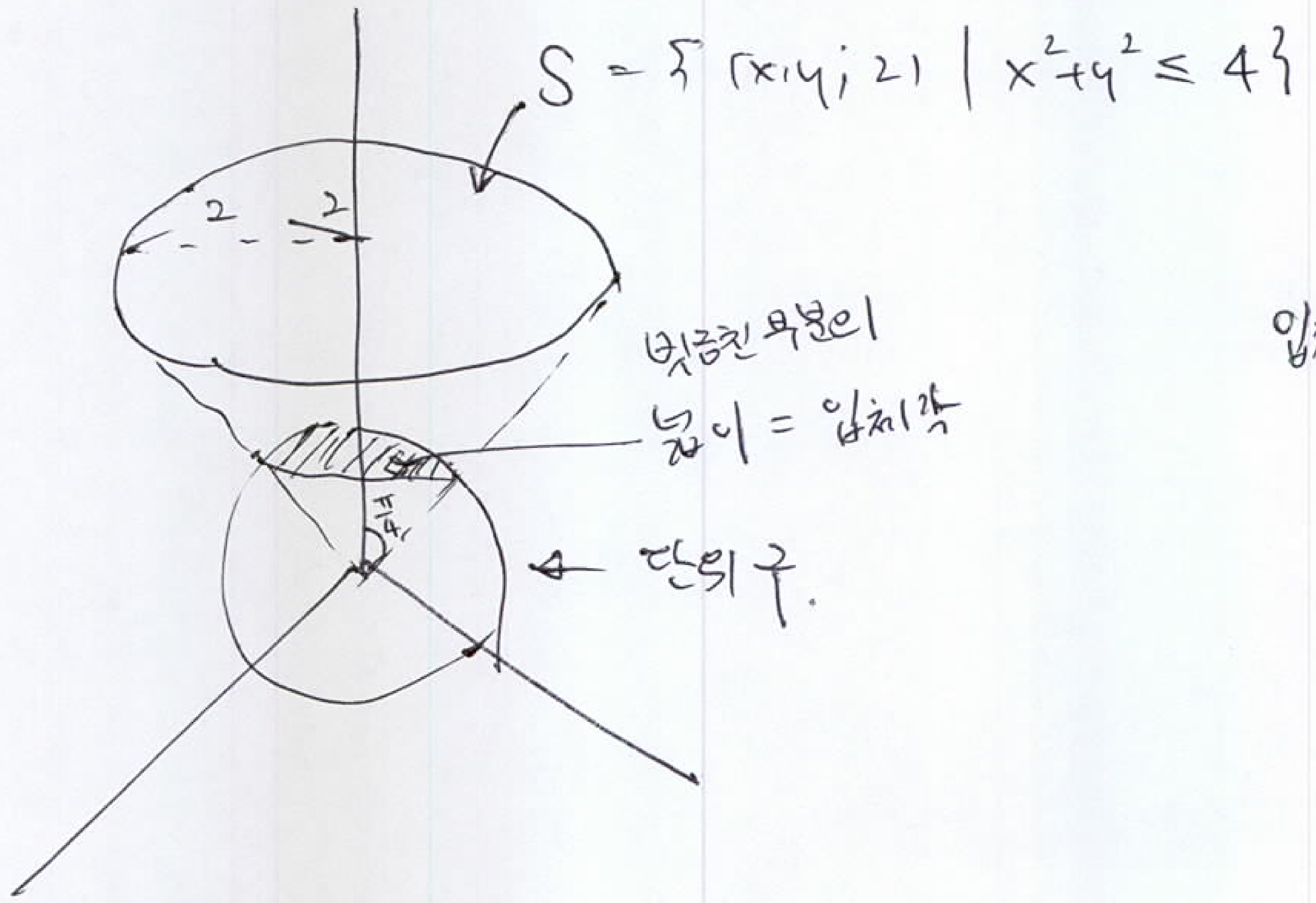
$$\bar{z} = \frac{1}{M}\iint_S zf\,dS = \frac{1}{12}\int_0^{\frac{\pi}{2}}\int_0^3 (x^2\cos\theta\sin\theta + \sin^2\theta)\,dx\,d\theta = \frac{1}{16}\pi + \frac{3}{8}$$

* 질량, 평균밀도, $\bar{x}, \bar{y}, \bar{z}$ 의 정의를 알고 있는가에 각각 2점씩 부여.

정확한 값에 각각 2점씩 부여

8.

(a)



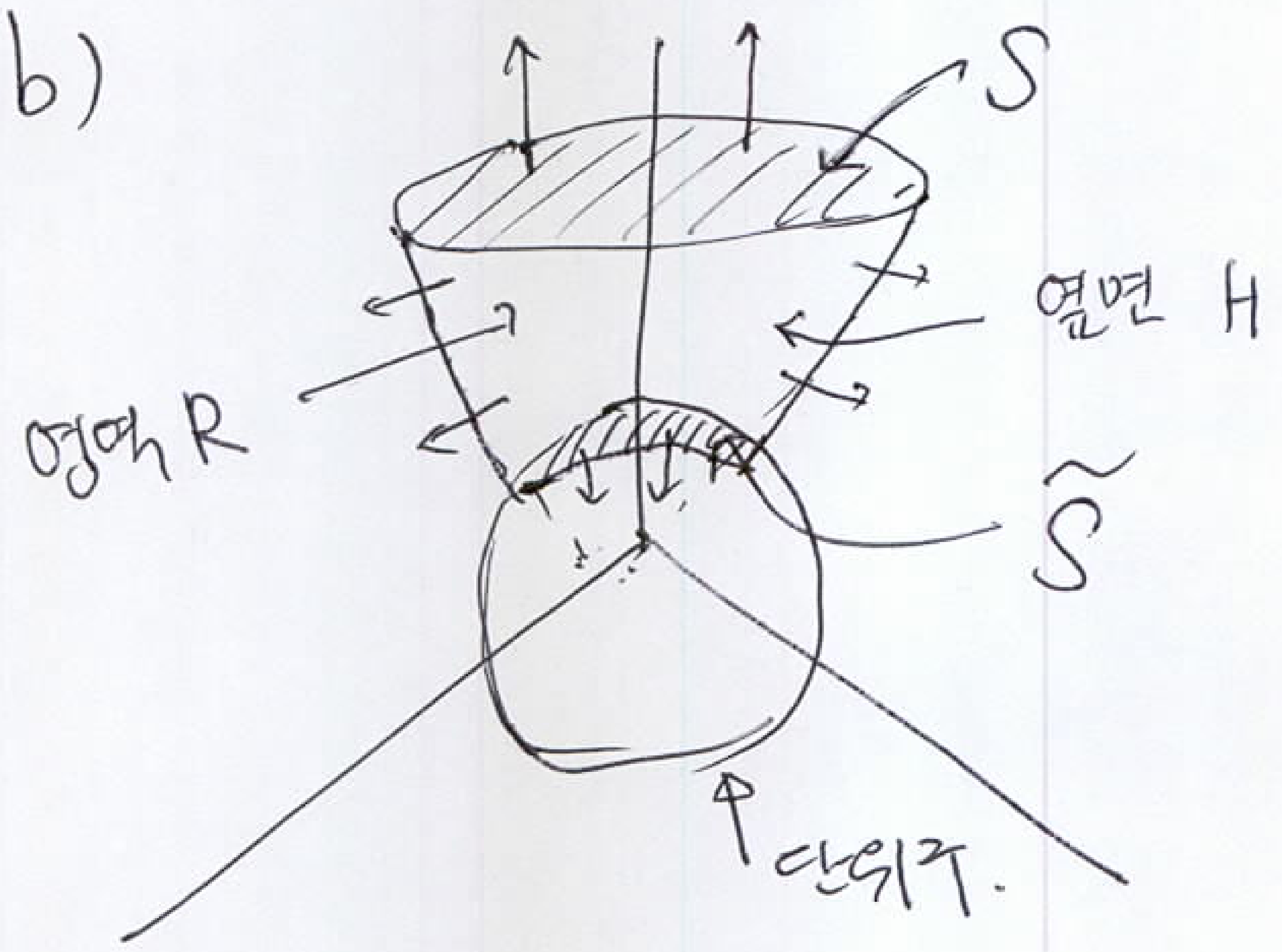
$$\text{입체각} = \int_0^{2\pi}\int_0^{\frac{\pi}{4}} 1\cdot \sin\varphi \, d\varphi \, d\theta$$ ┘ 10점

$$= 2\pi\left(1 - \frac{1}{\sqrt{2}}\right)$$

$$= \pi(2 - \sqrt{2})$$ ┘ 5점.

* 2π는 0점

(b)



그 두면, 영역 R의 경계는

$$\partial R = S \cup H \cup \tilde{S}$$

이때, $\tilde{S}$는 아래방향으로 향함이 주어짐.

$\operatorname{div} \mathbb{A} \equiv 0$ ┘ 5점. on $\mathbb{R}^3 \setminus \{0\}$ 이고, 발산정리에 의해

$$0 = \iiint_R \operatorname{div}\mathbb{A} \, dV = \iint_{\partial R} \mathbb{A}\cdot d\mathbf{S} = \iint_S \mathbb{A}\cdot d\mathbf{S} + \iint_H \mathbb{A}\cdot d\mathbf{S} + \iint_{\tilde{S}} \mathbb{A}\cdot d\mathbf{S}$$

이때, H의 법벡터와 $\mathbb{A}$는 수직으므로 $\iint_H \mathbb{A}\cdot d\mathbf{S} = 0$ 이다. ┘ 5점.

$$\therefore \iint_S \mathbb{A}\cdot d\mathbf{S} = -\iint_{\tilde{S}} \mathbb{A}\cdot d\mathbf{S} = -\iint_{\tilde{S}} \mathbb{A}\cdot \mathbf{n} \, dS = \iint_{\tilde{S}} 1 \, dS = \text{Area}(\tilde{S}) = \text{입체각}$$

$\mathbf{n} = (-x, -y, -z)$

$$= \pi(2 - \sqrt{2})$$ ┘ 5점.

9. $\operatorname{grad}\varphi(x,y,z) = (e^{-y} - ze^{-x},\ e^{-z} - xe^{-y},\ e^{-x} - ye^{-z})$

$\operatorname{curl}(\operatorname{grad}\varphi) = 0$ (직접계산 혹은 편미분 교환법칙)

+10

$C'(t) := (t, t, t)$ 라 하고 곡선 C와 C'을 연결한 폐곡선을 $\tilde{C}$라 하자. 이 때,

$$\int_C \operatorname{grad}\varphi \cdot ds + \left(-\int_{C'} \operatorname{grad}\varphi \cdot ds\right) = \int_{\tilde{C}} \operatorname{grad}\varphi \cdot ds$$

$$\underset{(\text{스토크스 정리})}{=} \iint_D \operatorname{curl}(\operatorname{grad}\varphi) \cdot dS$$

$$= 0$$

(D는 폐곡선 $\tilde{C}$에 의해 둘러싸인 영역이다.)

따라서,

$$\int_C \operatorname{grad}\varphi \cdot ds = \int_{C'} \operatorname{grad}\varphi \cdot ds$$

$$= \int_0^1 \operatorname{grad}\varphi(C'(t)) \cdot (1,1,1)\, dt$$

$$= \int_0^1 3(e^{-t} - te^{-t})\, dt$$

$$= 3e^{-1}$$

+10

* 스토크스정리를 적용하지 않고 답만 맞으면 5점.

* $\operatorname{curl}(\operatorname{grad}\varphi)$를 계산하여 0 임을 보이면 10점.

* 스토크스 정리를 적용하여 올바른 계산식을 세우고 답을 맞추면 10점.